Reflection over x-axis: $\begin{bmatrix}1&0\\0&-1\end{bmatrix}$ Reflection in origin: $\begin{bmatrix}-1&0\\0&-1\end{bmatrix}$ Reflection over y=x: $\begin{bmatrix}0&1\\1&0\end{bmatrix}$ Rotation: $\begin{bmatrix}\cos\theta&-\sin\theta\\ \sin\theta&\cos\theta\end{bmatrix}$ Scale by k: $\begin{bmatrix}k&0\\0&k\end{bmatrix}$

BASICS

system of linear equations (SLE):

$4x_1 - 7x_2 + 6x_3 = 5$
$1x_1 + 0x_2 + 9x_3 = 3$

coeff. matrix $\begin{bmatrix}4&-7&6\\1&0&9\end{bmatrix}$ augmented matrix $\begin{bmatrix}4&-7&6&5\\1&0&9&3\end{bmatrix}$

REF: $\begin{bmatrix}p&f&f\\0&p&f\\0&0&0\end{bmatrix}$ Zeros at bottom and below each pivot

RREF: $\begin{bmatrix}1&0&f\\0&1&0\\0&0&0\end{bmatrix}$ REF and pivots = 1 and all entries above and below = 0

Identity matrix (I): $\begin{bmatrix}1&\cdots&0\\ \vdots&\ddots&\vdots\\0&\cdots&1\end{bmatrix}$ $AI = A$, $IA = A$ "do nothing matrix"

solution set: Set of all possible solutions to an SLE. There are either 0, 1, or ∞ sln's.

Zero vector: $\vec{0} = \begin{bmatrix}0\\ \vdots\\0\end{bmatrix}$ also known as the trivial sol'n to $A\vec{x} = \vec{0}$

homogenous system: $A\vec{x} = \vec{0}$ has a non-trivial sol'n iff equation has 1+ free variables. If every column is a pivot col., just the trivial solution.

equivalent system: Obtained by replacing one ~~solution~~ system equation with a multiple of another. Same solution set.

$\frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix} = A^{-1}$ 2x2 Inverse

Multiplication properties:
$AB \neq BA$
$A^k = 0 \not\Rightarrow A = 0$
$AB = 0 \not\Rightarrow A = 0, B = 0$

matrix notation: Rows x Cols

non-homogenous system: $A\vec{x} = \vec{b}$ can have zero, one, ∞ solutions. Either one (trivial) or infinite solutions.

Transpose (A^T): Turn rows into columns. $A^{T^T} = A$

elementary row operations:
- Replacement: Replace row with sum of itself and multiple of another row.
- Interchange: Swap any two rows
- Scaling: Scale by a nonzero constant

linearly independent: not redundant, only trivial solution
linearly dependent: redundant, more than trivial solution
a single vector is L.I. if it's not $\vec{0}$. two vectors are L.I. as long as one is not a multiple of another.
L.I. vectors have max span. a set of vectors is L.D. if one is in the span of the others.

inconsistent system: no solution
$[\,0\;0\;\cdots\;0\;b\,]$ $b \neq 0$

unique solution: no non-pivot columns
infinite solutions: non-pivot columns (free variables)

vector: $\vec{x} = \begin{bmatrix}x_1\\x_2\\x_n\end{bmatrix}$ vectors can be scaled and added

more vectors than entries = L.D.
contains zero vector = L.D.

Pivot in every row: Rows span $\mathbb{R}^m$
Pivot in every column: Col's are L.I.

Inverse: Like the reciprocal but for matrices: $A(A^{-1}) = I$ $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$ $2\times2 \rightarrow$

Generally: Form $[A \mid I]$. Row reduce till left side becomes I, then A^{-1} is on right side. If not possible, then not invertible. $A \sim I$ iff columns of A are L.I.

Span$\{v_1, \ldots v_n\}$: All linear combinations of $v_1, v_2, \ldots, v_n$. Linear combinations are made by scaling and adding vectors.

$y + 5z = 0$
$2x + 5y + z = 0$
$-x + 2y - 6z = 0$
$\rightarrow$ Vector equation $x\begin{bmatrix}0\\2\\-1\end{bmatrix} + y\begin{bmatrix}1\\5\\2\end{bmatrix} + z\begin{bmatrix}5\\1\\-6\end{bmatrix} = \vec{0}$

matrix multiplication: $(Q \times R)(S \times T)$ — R and S must match; Q x T: dimensions of new matrix

TRANSFORMATIONS

$T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ — n: # col's in A (input); m: # rows in A (out)

$A\vec{x} = \vec{b}$ has a solution (and augmented matrix is consistent) iff $\vec{b}$ is a linear combination of the columns of A.

Domain: Space of input **Range:** Set of all outputs.
Codomain: Space of output **Example:** $\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$ collapses in xy-plane

All matrix transformations are linear transformations. $T(\vec{0}) = \vec{0}$ $T(c\vec{v}) = cT(\vec{v})$ $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ If we have a set that spans the domain, like I, then knowing what T does to that set tells us what T does to every vector.

Solve $A\vec{x} = \vec{b}$: Form augmented matrix $[A\;\vec{b}]$ and row reduce. If a pivot in every row, then columns of A span $\mathbb{R}^m$, each $\vec{b}$ in $\mathbb{R}^m$ is a linear combo of cols in A, and for each $\vec{b}$ in $\mathbb{R}^m$, $A\vec{x} = \vec{b}$ has a solution.
If pivot in every column solution is unique.

Columns of T: The columns of the transformation matrix T tell us what happens to $\vec{e}_1, \vec{e}_2, \vec{e}_n$.

$\begin{bmatrix}1&5\\2&-1\\0&3\end{bmatrix}$ this transforms from $\mathbb{R}^2$ to $\mathbb{R}^3$

Example: $T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}5x_1\\x_1+x_2\end{bmatrix}$ In other words, how does T transform the unit square?
$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}5\\1\end{bmatrix}$, $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix}$ $\rightarrow A = \begin{bmatrix}5&0\\1&1\end{bmatrix}$

Onto: If range = codomain, then onto.
One to one: If no info lost, then one to one.
Onto relates to span and one to one relates to linear independence. If $T(\vec{x}) = \vec{0}$ has only the trivial solution, then one to one.

Transformations, inverses: If T transforming a matrix A then T^{-1} undoes that transformation. If T collapses space, then T has no inverse.

INVERT. MATRIX THE^EM

- A is an invertible matrix.
- A is row-equiv. to I.
- A has n pivot positions.
- $A\vec{x} = \vec{0}$ has only trivial sol'n.
- Columns of A form L.I. set. $\det(A) \neq 0$
- Linear transform T ($\vec{x} \rightarrow A\vec{x}$) is one to one.
- $A\vec{x} = \vec{b}$ has at least one solution for each $\vec{b}$ in $\mathbb{R}^n$.
- Columns of A span $\mathbb{R}^n$.
- $T(x \rightarrow Ax)$ maps $\mathbb{R}^n$ to $\mathbb{R}^n$.
- A^T is an invertible matrix.
- $\exists$ nxn C such that $CA = I$
- " "D" " $DA = I$

in codomain / in domain

If one of these is true, then all are. If one is false, then all

Subspace: Any set H in $\mathbb{R}^n$ that contains the origin, preserves scalar multiplication, and preserves vector addition.

subspace / not a subspace

Subspaces of $\mathbb{R}^3$: Point (origin), line (through origin), plane (" "), whole space.

Column space: Col A is the span of the columns of A. Equiv. to range.

Null space: Set of all solutions to $A\vec{x} = \vec{0}$. Or all $\vec{x}$ that get mapped to $\vec{0}$ under A. Describes how a transformation matrix collapses space.

$\underset{x_1}{Mn_3O_9} + \underset{x_2}{H_3Cl_3} \rightarrow \underset{x_3}{Mn_3Cl_9} + \underset{x_4}{H_9O_3} + \underset{x_5}{Cl_9}$
Mn: $3x_1 - 3x_3 = 0$
O: $9x_1 - 3x_4 = 0$
H: $3x_2 - 9x_4 = 0$
Cl: $3x_2 - 9x_3 - 9x_5 = 0$
$\rightarrow \begin{bmatrix}3&0&-3&0&0&0\\9&0&0&-3&0&0\\0&3&0&-9&0&0\\0&3&-9&0&-9&0\end{bmatrix} \rightarrow$ whole nums
$x_1 = 1$, $x_2 = 9$, $x_3 = 1$, $x_4 = 3$, $x_5 = 2$

Orthogonal set: A set of vectors that are all mutually orthogonal. Orthogonality is the epitome of linear independence. Mutually orthogonal if all vectors dot to 0. If $S=\{\begin{bmatrix}3\\1\\1\end{bmatrix},\begin{bmatrix}-1\\2\\1\end{bmatrix},\begin{bmatrix}-\frac{1}{2}\\-2\\\frac{7}{2}\end{bmatrix}\}$ and is orthogonal set and $y=\begin{bmatrix}6\\1\\-8\end{bmatrix}$, then a linear combo of S, $y=\frac{y\cdot u_1}{u_1\cdot u_1}u_1+\frac{y\cdot u_2}{u_2\cdot u_2}u_2+\frac{y\cdot u_3}{u_3\cdot u_3}u_3=\frac{9}{11}u_1+\frac{1}{6}u_2+\frac{19}{33}u_3$. Ex 2: Show $u_1=\begin{bmatrix}1\\1\end{bmatrix}$, $u_2=\begin{bmatrix}-1\\1\end{bmatrix}$ are basis for $\mathbb{R}^2$. $u_1\cdot u_2=0$, so orthogonal. Two L.I. vectors in $\mathbb{R}^2$ form a basis. **Orthonormal set:** An orthogonal set of unit vectors. Standard basis vectors $\{e_1,e_2,e_3\}$ form an orthonormal set (unit vectors ✓ orthogonal ✓). If U is orthonormal, then U preserves lengths ($\|Ux\|=\|x\|$) and U preserves angles ($(Ux)\cdot(Uy)=x\cdot y$). And $(Ux)\cdot(Uy)=0$ iff $x\cdot y=0$.

sold by				
F	M	S		
0.1	0.1	0.2	F	
0.8	0.1	0.4	M	to
0.1	0.8	0.4	S	

$P_F=0.1P_F+0.1P_M+0.2P_S$
$P_M=0.8P_F+0.1P_M+0.4P_S$
$P_S=0.1P_F+0.8P_M+0.4P_S$
then transform to augmented matrix $=0$ and row reduce to solve

Basis of subspace: Smallest number of vectors that span a subspace. AKA minimal spanning set. L.I. Having a basis provides a coord. system for the space. **Dimension of subspace:** # of vectors in any basis for the subspace. Denoted dim H. **Rank:** # of pivot columns. **Rank-Nullity Theorem:** If A has n columns, rank A + dim Nul $A = n$. ie. # pivot columns + #non-pivot columns = #cols.

[↑ output, ← input] Suppose A is 3×4 matrix w/ rank=2. Then A transforms $\mathbb{R}^4$ into a 2D subspace of $\mathbb{R}^3$, collapsing two dimensions along the way. **Basis theorem:** Any set of p L.I. vectors will span p dimensions.

Invert Matrix Theorem Part 2.
If A is invertible, then..
- Columns of A form basis for $\mathbb{R}^n$
- Col $A=\mathbb{R}^n$
- dim Col $A=n$
- rank $A=n$
- Nul $A=\{\vec{0}\}$
- dim Nul $A=0$

Determinate: The determinate describes how a transformation changes the area of the unit square. If det(A)=25, then it increases the area of the unit square by 25. If det(A)=0, then collapsing space and not one to one, and not invertible.

$A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$, $\det(A)=ad-bc$ ← Determinate of 2×2. **Determinate of 3×3:** Choose a row to be cofactors (hopefully w/ zeros), apply signs $\begin{bmatrix}+&-&+\\-&+&-\\+&-&+\end{bmatrix}$, and find ~~cofactor~~ determinate of 2×2 matrices within.

$\begin{bmatrix}1&2&3\\4&3&2\\3&2&1\end{bmatrix}=1\left(\begin{vmatrix}3&2\\2&1\end{vmatrix}\right)-2\left(\begin{vmatrix}4&2\\3&1\end{vmatrix}\right)+3\left(\begin{vmatrix}4&3\\3&2\end{vmatrix}\right)=-1+4-3=0$

Unit vector: Vector w/ length=1. To find, divide each vector component by its magnitude.

Dot (inner) product: Produces a scalar. Dims must match. $\begin{bmatrix}2\\-1\\7\end{bmatrix}\cdot\begin{bmatrix}2\\0\\5\end{bmatrix}=39$
$u\cdot v=v\cdot u$
$(u+v)\cdot w=u\cdot w+v\cdot w$
$(cu)\cdot v=c(u\cdot v)=u\cdot(cv)$
$u\cdot u\geq0$ and $u\cdot u=0$ iff $u=0$

Distance: $\text{dist}(u,v)=\|u-v\|$

Orthogonality: Two vectors u v are orthogonal iff $u\cdot v=0$. The $\vec{0}$ is orthogonal to every vector.

Length of a vector: $\|v\|=\sqrt{v\cdot v}$, $\|v\|^2=v\cdot v$ (basically just the Euclidean distance form.) $\vec{v}=\begin{bmatrix}3\\2\\1\end{bmatrix}$ $\|v\|=\sqrt{3^2+2^2+1^2}=\sqrt{14}$ As unit vector: $\langle\frac{3}{\sqrt{14}},\frac{2}{\sqrt{14}},\frac{1}{\sqrt{14}}\rangle$

Orthogonal complement: If a vector z is orthogonal to every vector in a subspace W, then z is orthogonal to W. The set of all vectors z that are orthogonal to W is called the orthogonal complement of W, and is denoted $W^\perp$ ("W perp"). ↖ If orthogonal to the basis of a subspace, then orthogonal to every vector in that subspace.

Triangular matrix: $\begin{bmatrix}*&*\\0&*\end{bmatrix}$ or $\begin{bmatrix}*&0\\ *&*\end{bmatrix}$ All zeros below main diag, above, or both. Product along main diagonal is determinate. But row operations change it. Interchange: swaps sign (each time). Scaling: Scales det. (so mult. by reciprocal to undo). Replacement: No change.

Row space: Set of all linear combinations of row vectors of A. Row A = Col A^T In domain.

Coordinate systems: Coefficients of basis vectors define our coordinates. Impossible to have coordinates w/o choosing a basis. A different basis means a different matrix for the same transformation.

Orthogonal matrices are isometries, ie. reflections and rotations. $U^{-1}=U^T$

Change of basis matrix: Transforms one set of coords into another. P_B represents a transformation that takes standard basis to B basis. But as coordinate change it takes B coordinates to standard coordinates.

$b_1=4\vec{c}_1+\vec{c}_2$, $b_2=-6\vec{c}_1+\vec{c}_2$ → $\begin{bmatrix}4&-6\\1&1\end{bmatrix}$ change of basis matrix $C\leftarrow B$
$\vec{x}=3\vec{b}_1+\vec{b}_2$ → $[\vec{x}]_B=\begin{bmatrix}3\\1\end{bmatrix}$
$[\vec{x}]_C=[3\vec{b}_1+\vec{b}_2]_C=3[\vec{b}_1]_C+1[\vec{b}_2]_C=[[\vec{b}_1]_C[\vec{b}_2]_C]\begin{bmatrix}3\\1\end{bmatrix}$
$=\begin{bmatrix}4&-6\\1&1\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}=\begin{bmatrix}6\\4\end{bmatrix}$ ← $[x]_C$ To go from $B\leftarrow C$ we can just take the inverse of $C\leftarrow B$ matrix.

Eigenvector: A vector that, when transformed by a transformation matrix T, remains on the same line (is only scaled). Ex. In $\mathbb{R}^2$, a reflection over $y=x$ preserves vectors along line $y=x$ (scales by 1) and line $y=-x$ (scales by −1). **Eigenvalue:** The factor that a given eigenvector is scaled by. λ

Eigenspace: The set of all eigenvectors associated with a particular eigenvalue.

Given $A=\begin{bmatrix}-5&2\\-7&4\end{bmatrix}$ compute $A\vec{x}$ for $\vec{x}=\begin{bmatrix}1\\1\end{bmatrix}$ → $\begin{bmatrix}-5&2\\-7&4\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}-3\\-3\end{bmatrix}=-3\begin{bmatrix}1\\1\end{bmatrix}$. So $\begin{bmatrix}-3\\-3\end{bmatrix}$ is an eigenvector w/ eigenvalue $\lambda=-3$. Basically we can factor out −3 to get back to $\vec{x}$.

$\vec{0}$ cannot be an eigenvector, but 0 can be an eigenvalue. $\lambda=0$ implies A not invertible, det(A)=0, etc. Given ~~n×A~~ n×n matrix A, A has at least one eigenvalue and thus at least one eigenvector. At most n eigenvalues and n L.I. ~~vects~~ eigenvectors. $\lambda I=\begin{bmatrix}\lambda&0\\0&\lambda\end{bmatrix}$

Find eigenvalues: Solve $\det(\lambda I-A)=0$ for λ ← this becomes the characteristic polynomial

Find eigenvectors: Plug in a particular eigenvalue for λ and solve $(\lambda I-A)\vec{x}=0$ (i.e. augmented matrix equal to $\vec{0}$, RREF, etc. Concluding statement might be: Eigenspace for $\lambda=2$ is span$\{\begin{bmatrix}2\\7\end{bmatrix}\}$

Diagonal matrices: Zeros in every entry except those in the main diagonal. $\begin{bmatrix}2&0&0\\0&-3&0\\0&0&1\end{bmatrix}$ Eigenvectors are standard basis vectors. ie $\begin{bmatrix}1\\0\\0\end{bmatrix}\xrightarrow{A}\begin{bmatrix}2\\0\\0\end{bmatrix}$, $\lambda=2$. $D^k=\begin{bmatrix}2^k&0&0\\0&-3^k&0\\0&0&1^k\end{bmatrix}$

$A=PDP^{-1}$
Ex: $\underset{A}{\begin{bmatrix}-2&12\\-1&5\end{bmatrix}}=\underset{P}{\begin{bmatrix}3&4\\1&1\end{bmatrix}}\underset{D}{\begin{bmatrix}2&0\\0&1\end{bmatrix}}\underset{P^{-1}}{\begin{bmatrix}-1&4\\1&-3\end{bmatrix}}$ ← $E_2=\text{Span}\{\begin{bmatrix}3\\1\end{bmatrix}\}$, $E_1=\text{Span}\{\begin{bmatrix}4\\1\end{bmatrix}\}$
Eigenvectors: $\{\begin{bmatrix}3\\1\end{bmatrix},\begin{bmatrix}4\\1\end{bmatrix}\}$ order matters!
Eigenvalues: $\lambda=2$ $\lambda=1$

An n×n matrix with n distinct eigenvals is diagonalizable. The eigenvalues of a triangular matrix are numbers on main diagonal. Eigenspaces are linearly independent.

In order for A to be diagonalizable, dimensions of eigenspaces must sum to n (w/ n×n matrix).

Determinate: So-called because determines if solution exists. If det(A) ≠ 0 then A is invertible and a unique solution exists to $A\vec{x}=\vec{b}$ for every $\vec{b}$. $\det(AB)=\det(A)\det(B)$

To be a basis, set must be L.I. and must be a spanning set.

From ~~standard~~ basis B to standard: $B=\{\begin{bmatrix}2\\-1\end{bmatrix},\begin{bmatrix}-2\\0\end{bmatrix}\}$ $[\vec{x}]_B=\begin{bmatrix}8\\10\end{bmatrix}$ $\vec{x}=8\begin{bmatrix}2\\-1\end{bmatrix}+10\begin{bmatrix}-2\\0\end{bmatrix}=\begin{bmatrix}-4\\-8\end{bmatrix}$

Orthogonal projection of $\vec{y}$ onto $\vec{u}$: $\frac{y\cdot u}{u\cdot u}\vec{u}=\hat{y}$. For distance between point and line take magnitude of $y-\hat{y}$, or